Θέμα 49

Α. $f^3(x)+f(x)=x^3\ ,\ x\in\mathbb{R}$

Θεωρώ $\phi(x)=x^3+x\ ,\ x\in\mathbb{R}$

$\phi'(x)=3x^2+1>0 \Rightarrow \phi \uparrow$

Έχουμε $(-x)^3=-x^3 \Leftrightarrow$

$f^3(-x)+f(-x)=-f^3(x)-f(x)$

$\phi(f(-x))=\phi(-f(x)) \overset{1-1}{\Rightarrow}$

$f(-x)=-f(x)$, f περιττή

Β1. $x_1<x_2 \Leftrightarrow x_1^3<x_2^3 \Leftrightarrow f^3(x_1)+f(x_1)<f^3(x_2)+f(x_2)$

$\Leftrightarrow \phi(f(x_1))<\phi(f(x_2)) \Leftrightarrow f(x_1)<f(x_2)$ οπότε $f\uparrow$

Β2. $f(x)+2021x=0$

Θεωρώ $g(x)=f(x)+2021x\ ,\ x\in\mathbb{R}$

$g'(x)=f'(x)+2021>0$

$\Rightarrow g\uparrow$

$g(0)=f(0)+0=0$

Άρα $g\uparrow$ θα έχει μοναδική λύση το 0

Γ1. Έστω $f(x)>x+1$ τότε $f^3(x)>(x+1)^3 \Leftrightarrow \left.\begin{array}{l} f^3(x)>x^3+3x^2+3x+1 \\ f(x)>x+1 \end{array}\right\} \overset{+}{\Rightarrow}$

$\Rightarrow \cancel{x^3}>\cancel{x^3}+3x^2+4x+2 \Leftrightarrow \left.\begin{array}{l} 3x^2+4x+2<0 \\ \Delta=16-24=-8\ ,\ \alpha=3 \end{array}\right\}$ Άτοπο

Άρα $f(x)\le x+1$

Γ2. Έστω $f(x)<x-1$ τότε $f^3(x)<(x-1)^3 \Leftrightarrow \left.\begin{array}{l} f^3(x)<x^3-3x^2+3x-1 \\ f(x)<x-1 \end{array}\right\} \overset{+}{\Rightarrow}$

$\Rightarrow \cancel{x^3}<\cancel{x^3}-3x^2+4x-2 \Rightarrow \left.\begin{array}{l} -3x^2+4x-2>0 \\ \Delta=16-24=-8\ ,\ \alpha=-3 \end{array}\right\}$ Άτοπο

Άρα $f(x)\ge x-1$

Γ3. $\left.\begin{array}{l} f(x)\le x+1 \\ \lim\limits_{x\to-\infty}(x+1)=-\infty \end{array}\right\} \Rightarrow \lim\limits_{x\to-\infty} f(x)=-\infty$ $\qquad \left.\begin{array}{l} f(x)\ge x-1 \\ \lim\limits_{x\to+\infty}(x-1)=+\infty \end{array}\right\} \Rightarrow \lim\limits_{x\to+\infty} f(x)=+\infty$

Άρα $f(\mathbb{R})=\left(\lim\limits_{x\to-\infty} f(x), \lim\limits_{x\to+\infty} f(x)\right)=(-\infty,+\infty)$

Γ4. $$\frac{\frac{1}{f(x)}}{1+\frac{x}{f(x)}+\left(\frac{x}{f(x)}\right)^2}=\frac{\frac{1}{\cancel{f(x)}}}{\frac{f^2(x)+xf(x)+x^2}{f^2(x)}}=\frac{f(x)}{f^2(x)+xf(x)+x^2}=\frac{f(x)(f(x)-x)}{(f^2(x)+xf(x)+x^2)(f(x)-x)}=$$

$$=\frac{\cancel{f(x)}(f(x)-x)}{\cancel{f^3(x)-x^3}}=-f(x)+x=-(f(x)-x)$$

Γ5. $x-1\le f(x)\le x+1 \underset{x>0}{\overset{x\ne0}{\Leftrightarrow}} \left.\begin{array}{l} 1-\frac{1}{x}\le\frac{f(x)}{x}\le 1+\frac{1}{x} \\ \lim\limits_{x\to+\infty}\left(1-\frac{1}{x}\right)=1 \text{ και } \lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)=1 \end{array}\right\} \overset{\text{Κ.Π.}}{\Longrightarrow} \lim\limits_{x\to+\infty}\frac{f(x)}{x}=1$

Γ6. $$\lim_{x\to+\infty}(f(x)-x)=\lim_{x\to+\infty}\left(-\frac{\frac{1}{f(x)}}{1+\frac{x}{f(x)}+\left(\frac{x}{f(x)}\right)^2}\right)=-\frac{0}{1+1+1}=\frac{0}{-3}=0$$

Άρα η $y=x$ ασύμπτωτη της C_f στο $+\infty$