Θέμα 42

Α. $h(x)=\dfrac{1}{x^2\sqrt{x^2+1}}$, $x>0$

$h'(x)=-\dfrac{1}{x^4(x^2+1)}\cdot\left(2x\cdot\sqrt{x^2+1}+x^2\cdot\dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\right)=$

$=-\dfrac{1}{x^3(x^2+1)}\cdot\left(\dfrac{2x^2+2+x^2}{\sqrt{x^2+1}}\right)=-\dfrac{3x^2+2}{x^3(x^2+1)\sqrt{x^2+1}}<0$

Άρα $h \downarrow$

$h(D)=\left(\lim\limits_{x\to+\infty}h(x),\lim\limits_{x\to0^+}h(x)\right)=$

$=(0,+\infty)$ αφού

$\lim\limits_{x\to+\infty}h(x)=\lim\limits_{x\to+\infty}\dfrac{1}{x^2\sqrt{x^2+1}}=0$ και $\lim\limits_{x\to0^+}\dfrac{1}{x^2\cdot\sqrt{x^2+1}}=+\infty$

Β. $f'(x)=h(x)+\dfrac{1}{x^2}\cdot h\left(\dfrac{1}{x}\right)=\dfrac{1}{x^2\cdot\sqrt{x^2+1}}+\dfrac{1}{\cancel{x^2}}\cdot\dfrac{\cancel{x^2}}{\sqrt{\frac{1}{x^2}+1}}=\dfrac{1}{x^2\sqrt{x^2+1}}+\dfrac{x}{\sqrt{x^2+1}}=\dfrac{1+x^3}{x^2\sqrt{x^2+1}}$

$g'(x)=x^3\cdot h(x)+\dfrac{1}{x^5}h\left(\dfrac{1}{x}\right)=x^{\cancel{3}}\cdot\dfrac{1}{\cancel{x^2}\sqrt{x^2+1}}+\dfrac{1}{x^{\cancel{5}\,3}}\dfrac{\cancel{x^2}}{\sqrt{\frac{1}{x^2}+1}}=\dfrac{x}{\sqrt{x^2+1}}+\dfrac{1}{x^2\sqrt{x^2+1}}=\dfrac{x^3+1}{x^2\sqrt{x^2+1}}$

Γ. $f'(x)=\dfrac{x^3+1}{x^2\sqrt{x^2+1}}>0$

$f(x)=\left(1-\dfrac{1}{x}\right)\sqrt{x^2+1}+c$

$f(x)-\left(1-\dfrac{1}{x}\right)\sqrt{x^2+1}=c$

Θεωρώ $\phi(x)=f(x)-\left(1-\dfrac{1}{x}\right)\sqrt{x^2+1}$

$\phi'(x)=f'(x)-\dfrac{1}{x^2}\sqrt{x^2+1}-\left(1-\dfrac{1}{x}\right)\cdot\dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}=$

$=f'(x)-\dfrac{\sqrt{x^2+1}}{x^2}-\dfrac{x}{\sqrt{x^2+1}}+\dfrac{1}{\sqrt{x^2+1}}=$

$=\dfrac{x^3+1}{x^2\sqrt{x^2+1}}-\dfrac{\sqrt{x^2+1}}{x^2}+\dfrac{1-x}{\sqrt{x^2+1}}=\dfrac{\cancel{x^3}+\cancel{1}-\cancel{x^2}-\cancel{1}+\cancel{x^2}-\cancel{x^3}}{x^2\sqrt{x^2+1}}=0$

Άρα $\phi'(x)=0$ οπότε $\phi(x)$ σταθερή δηλ. $\phi(x)=c \Leftrightarrow f(x)=\left(1-\dfrac{1}{x}\right)\sqrt{x^2+1}+c$

Δ. $f'(x)=g'(x) \Leftrightarrow f(x)=g(x)+c_1$

Αφού C_f και C_g έχουν τουλάχιστον 1 κοινό σημείο

για $x=x_0$: $f(x_0)=g(x_0)$

Άρα $\cancel{f(x_0)}=\cancel{g(x_0)}+c_1 \Leftrightarrow c_1=0$

οπότε $f(x)=g(x)$ άρα $c=0$ και $m=1$