Θέμα 37

Α1. f_1 συνεχής άρα $f_1(1) = \lim_{x\to 1} f_1(x) = \lim_{x\to 1} \frac{x-1}{x\ln x} \overset{DLH}{=} \lim_{x\to 1} \frac{1}{\ln x + 1} = 1$

Α2. $x^x = e^{x-1} \Leftrightarrow \ln x^x = \ln e^{x-1} \Leftrightarrow x\cdot \ln x = (x-1)\cdot \cancel{\ln e}^{1} \Leftrightarrow \frac{x-1}{x\ln x} = 1 \Leftrightarrow f_1(x) = f_1(1)$

$f_1'(x) = \frac{(x-1)'\, x\ln x - (x-1)\cdot (x\ln x)'}{(x\ln x)^2} = \frac{x\ln x - (x-1)(\ln x + 1)}{(x\ln x)^2} = \frac{\cancel{x\ln x} - \cancel{x\ln x} - x + \ln x + 1}{(x\ln x)^2}$

$= \frac{\ln x + 1 - x}{(x\ln x)^2}$ ξέρουμε $\ln x \leq x - 1 \Leftrightarrow \ln x + 1 - x \leq 0$

οπότε $f_1 \searrow$ άρα 1-1

άρα $f_1(x) = f_1(1) \Leftrightarrow x = 1$

Β. $D_f = \begin{cases} x \in D_{f_2} \\ f_2(x) \in D_{f_1} \end{cases} = \begin{cases} x \in \mathbb{R} \\ e^x + 1 \neq 1 \text{ και } e^x + 1 > 1 \text{ και } 0 < e^x + 1 < 1 \\ e^x \neq 0 \qquad e^x > 0 \end{cases} = \mathbb{R}$

$f(x) = (f_1 \circ f_2)(x) = f_1(f_2(x)) = f_1(e^x + 1) = \frac{e^x + \cancel{1} - \cancel{1}}{(e^x+1)\ln(e^x+1)} = \frac{e^x}{(e^x+1)\ln(e^x+1)}$

Γ1. $g(x) = \ln(e^x+1) - e^x,\ x \in \mathbb{R}$

$g'(x) = \frac{1}{e^x+1}\cdot e^x - e^x = e^x\cdot\left(\frac{1}{e^x+1} - 1\right) = e^x\cdot\left(\frac{\cancel{1} - e^x - \cancel{1}}{e^x+1}\right) = \frac{-(e^x)^2}{e^x+1} < 0 \Rightarrow g \searrow$

$g(\mathbb{R}) = \left(\lim_{x\to+\infty} g(x), \lim_{x\to-\infty} g(x)\right) = (-\infty, 0)$ αφού

$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty}\left(\ln(e^x+1) - e^x\right) = \lim_{x\to+\infty} e^x\left(\frac{\ln(e^x+1)}{e^x} - 1\right) = \lim_{x\to+\infty} e^x\cdot\left(\frac{\frac{e^x}{e^x+1}}{e^x} - 1\right) = (+\infty)(-1) = -\infty$

$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty}\left(\ln(e^x+1) - e^x\right) = 0 - 0 = 0$

Γ2. $f'(x) = \frac{e^x\cdot(e^x+1)\ln(e^x+1) - e^x\cdot e^x \ln(e^x+1) - e^x\cdot\cancel{(e^x+1)}\frac{1}{\cancel{e^x+1}}\cdot e^x}{(e^x+1)^2\cdot\ln^2(e^x+1)} = \frac{e^x\cdot\left(\ln(e^x+1) - e^x\right)}{(e^x+1)^2\cdot\ln^2(e^x+1)} =$

$= \frac{e^x\cdot g(x)}{(e^x+1)^2\cdot\ln^2(e^x+1)}$ άρα το πρόσημο της f' καθορίζεται από το πρόσημο της g, όμως $g(x) < 0$ άρα $f'(x) < 0$

Γ3. $f'(x) < 0 \Rightarrow f \searrow$ $\quad f(\mathbb{R}) = \left(\lim_{x\to+\infty} f(x), \lim_{x\to-\infty} f(x)\right) = (0, 1)$ αφού

$\lim_{x\to+\infty} f(x) \overset{DLH}{=} \lim_{x\to+\infty} \frac{e^x}{e^x\ln(e^x+1) + e^x} = 0$ και $\lim_{x\to-\infty} f(x) \overset{DLH}{=} \lim_{x\to-\infty} \frac{e^x}{e^x\ln(e^x+1) + e^x} = 1$

Δ. $\ln 4\, f(x) e^{f(x)} = e^{\frac{1}{\ln 4}} \Leftrightarrow \ln\left(\ln 4\cdot f(x)\cdot e^{f(x)}\right) = \ln e^{\frac{1}{\ln 4}} \Leftrightarrow \ln(\ln 4) + \ln f(x) + f(x) = \frac{1}{\ln 4}$

$\Leftrightarrow \ln f(x) + f(x) = \ln\left(\frac{1}{\ln 4}\right) + \frac{1}{\ln 4}$

Θεωρώ $h(x) = \ln x + x,\ x > 0$

$h'(x) = \frac{1}{x} + 1 > 0 \Rightarrow h \nearrow$

$\Rightarrow h(f(x)) = h\left(\frac{1}{\ln 4}\right) \Leftrightarrow f(x) = \frac{1}{\ln 4} \Leftrightarrow$

$\Leftrightarrow f(x) = f(0) \Leftrightarrow x = 0$