Θέμα 36

Α1. $\lim_{x\to 0}\left(\frac{g(x)}{x}\right)^2+\left(g''(0)-2\right)^2=0 \Leftrightarrow \lim_{x\to 0}\frac{g(x)}{x}=0$ και $g''(0)-2=0$

$g(0)=\lim_{x\to 0}g(x)=\lim_{x\to 0}x\cdot\frac{g(x)}{x}=\lim_{x\to 0}x\cdot\lim_{x\to 0}\frac{g(x)}{x}=0 \Leftrightarrow g(0)=0$

$\lim_{x\to 0}\frac{g(x)}{x}=\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}=g'(0) \Leftrightarrow g'(0)=0$

Α2. $g''(0)-2=0 \Leftrightarrow g''(0)=2$

Α3. $|g''(x)-g''(x_0)|\le (x-x_0)^2=|x-x_0|^2 \Leftrightarrow \left|\frac{g''(x)-g''(x_0)}{x-x_0}\right|\le |x-x_0|$

$$\left.\begin{array}{l} \Leftrightarrow -|x-x_0|\le \frac{g''(x)-g''(x_0)}{x-x_0}\le |x-x_0| \\ \lim_{x\to x_0}\left(-|x-x_0|\right)=0 \text{ και } \lim_{x\to x_0}\left(|x-x_0|\right)=0 \end{array}\right\} \overset{\text{Κ.Π}}{\Rightarrow} \lim_{x\to x_0}\frac{g''(x)-g''(x_0)}{x-x_0}=0 \text{ Άρα } g'''(x)=0$$

τότε $\left.\begin{array}{l} g''(x)=c \\ g''(0)=2 \end{array}\right\} \Rightarrow g''(x)=2$ τότε $\left.\begin{array}{l} g'(x)=2x+c \\ g'(0)=c \end{array}\right\} \overset{c=0}{\Rightarrow} g'(x)=2x$

τότε $\left.\begin{array}{l} g(x)=x^2+c \\ g(0)=0 \end{array}\right\} \overset{c=0}{\Rightarrow} g(x)=x^2,\ x\ge 0$

Β1. $(1+g(x))f'(x)=x$ για $x=0$ $\quad 1\cdot f'(0)=0 \Leftrightarrow f'(0)=0$

Β2. $(1+g(x))\cdot f'(x)=x \Leftrightarrow (1+x^2)f'(x)=x \overset{x^2+1\neq 0}{\Longleftrightarrow} f'(x)=\frac{x}{x^2+1} \Leftrightarrow f'(x)=\left(\frac{1}{2}\ln(x^2+1)\right)'$

Άρα $\left.\begin{array}{l} f(x)=\ln\sqrt{x^2+1}+c \\ f(0)=0 \end{array}\right\} \overset{c=0}{\Rightarrow} f(x)=\ln\sqrt{x^2+1},\ x\ge 0$

$\lim_{x\to 0}\frac{2(e^x-1)\cdot f(2x)-xf(x)}{e^x-1}=0 \Leftrightarrow \lim_{x\to 0}\frac{2e^xf(2x)+2(e^x-1)\cdot 2f'(2x)-f(x)-xf'(x)}{e^x}=0 \Leftrightarrow 2f(0)-f(0)=0$

$\Leftrightarrow f(0)=0$ Άρα $f(x)=\ln\sqrt{x^2+1},\ x\ge 0$

Β3. $f'(x)\le\frac{1}{2} \Leftrightarrow \frac{x}{x^2+1}\le\frac{1}{2} \Leftrightarrow x^2+1\ge 2x \Leftrightarrow (x-1)^2\ge 0$ Προφανές

Β4. Αν $\alpha=\beta$ Προφανές

Αν $\alpha<\beta$ ΘΜΤ $[\alpha,\beta]$ έχουμε $\xi\in(\alpha,\beta)$ τέτοιο ώστε $f'(\xi)=\frac{f(\beta)-f(\alpha)}{\beta-\alpha}$

$2f(\beta)-2f(\alpha)\le\beta-\alpha \Leftrightarrow \frac{f(\beta)-f(\alpha)}{\beta-\alpha}\le\frac{1}{2} \Leftrightarrow f'(\xi)\le\frac{1}{2}$ Προφανές

Β5. $f'(\xi)\le\frac{1}{2} \Leftrightarrow \frac{1}{2}\,\frac{\ln(\beta^2+1)-\ln(\alpha^2+1)}{\beta-\alpha}\le\frac{1}{2} \Leftrightarrow \ln(\beta^2+1)-\ln(\alpha^2+1)\le\beta-\alpha \Leftrightarrow$

$\Leftrightarrow \alpha-\ln(\alpha^2+1)\le\beta-\ln(\beta^2+1) \Leftrightarrow \ln\frac{e^\alpha}{\alpha^2+1}\le\ln\frac{e^\beta}{\beta^2+1} \Leftrightarrow \frac{e^\alpha}{\alpha^2+1}\le\frac{e^\beta}{\beta^2+1} \Leftrightarrow (\beta^2+1)e^\alpha\le(\alpha^2+1)e^\beta$

Γ. $f(x)=g(x) \Leftrightarrow f(x)-g(x)=0$, Θεωρώ $h(x)=f(x)-g(x)$

$h'(x)=f'(x)+g'(x)=-\frac{x(2x^2+1)}{x^2+1}$

x	$-\infty$	0	$+\infty$
h'	//////	0	$-$
h	//////		↘

Άρα μοναδική ρίζα $x=0$

επίσης $f(0)=g(0)=0$

$f'(0)=g'(0)=0$