Θέμα 3ο

Α1. $f'(0)=1 \Leftrightarrow \lim_{h\to 0}\frac{f(h+0)-f(0)}{h}=1 \Leftrightarrow \lim_{h\to 0}\frac{f(h)-f(0)}{h}=1$

$f(x+h)=f(x)+f(h)+xh$ για $x=h=0$ έχουμε $\cancel{f(0)}=\cancel{f(0)}+f(0) \Leftrightarrow f(0)=0$

$\left.\right\} \lim_{h\to 0}\frac{f(h)}{h}=1$

Α2. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\cancel{f(x)}+f(h)+xh-\cancel{f(x)}}{h}=\lim_{h\to 0}\left(\frac{f(h)}{h}+\frac{x\cancel{h}}{\cancel{h}}\right)=1+x$

Άρα $f'(x)=x+1$

Α3. $f'(x)=x+1$

$\left.\begin{array}{l} f(x)=\frac{1}{2}x^2+x+c \\ \text{όμως } f(0)=0 \end{array}\right\} \Rightarrow c=0$ Άρα $f(x)=\frac{1}{2}x^2+x$

Β. $\varepsilon_1: y-f(x_1)=f'(x_1)(x-x_1)$

$\varepsilon_2: y-f(x_2)=f'(x_2)(x-x_2)$

$\varepsilon_1 \perp \varepsilon_2 \Rightarrow f'(x_1)\cdot f'(x_2)=-1$

$(x_1+1)\cdot(x_2+1)=-1$

$\boxed{x_1x_2+x_1+x_2=-2}$

Οι $\varepsilon_1, \varepsilon_2$ τέμνονται στο Μ

Άρα $\begin{cases} y=f'(x_1)x-f'(x_1)\cdot x_1+f(x_1) \Leftrightarrow y=(x_1+1)\cdot x-(x_1+1)\cdot x_1+\frac{1}{2}x_1^2+x_1 \\ y=f'(x_2)\cdot x-f'(x_2)x_2+f(x_2) \end{cases}$

$\Leftrightarrow \begin{cases} -(x_1+1)\cdot x+y=-\frac{1}{2}x_1^2 \\ -(x_2+1)\cdot x+y=-\frac{1}{2}x_2^2 \end{cases}$

$D=\begin{vmatrix} -(x_1+1) & 1 \\ -(x_2+1) & 1 \end{vmatrix}=-x_1-1+x_2+1=x_2-x_1$

$D_y=\begin{vmatrix} -(x_1+1) & -\frac{1}{2}x_1^2 \\ -(x_2+1) & -\frac{1}{2}x_2^2 \end{vmatrix}=\frac{1}{2}x_1x_2^2+\frac{1}{2}x_2^2-\frac{1}{2}x_1^2x_2-\frac{1}{2}\ldots$

$=\frac{1}{2}(x_1x_2^2-x_1^2x_2+x_2^2-x_1^2)$

$=\frac{1}{2}(x_1x_2(x_2-x_1)+x_2^2-x_1^2)$

$=\frac{1}{2}(x_2-x_1)(x_1x_2+x_2+x_1\ldots$

$=-(x_2-x_1)$

Άρα $y=\frac{D_y}{D}=\frac{x_2-x_1}{-(x_2-x_1)}=-1$

Οπότε $c=-1$ Άρα $\varepsilon: y=-1$

Γ. $(e^x+x-1)h(x)=2f(x)$

$e^x+x-1\neq 0 \Leftrightarrow x\neq 0$ τότε $h(x)=\frac{x^2+2x}{e^x+x-1}$

$g(x)=e^x+x-1$

$g'(x)=e^x+1>0$

Άρα $g \nearrow$

$e^x+x-1=0 \Leftrightarrow x=0$ τότε $h(0)=\lim_{x\to 0}\frac{x^2+2x}{e^x+x-1}\overset{DLH}{=}\lim_{x\to 0}\frac{2x+2}{e^x+1}=\frac{2}{2}=1$

$h(x)=\begin{cases} \frac{x^2+2x}{e^x+x-1}, & x\neq 0 \\ 1, & x=0 \end{cases}$